1.7 Mode Theory for Cylindrical Waveguide

- To analyze the optical fiber propagation mechanism within a fiber, Maxwell equations are to solve subject to the cylindrical boundary conditions at core-cladding interface. The core-cladding boundary conditions lead to coupling of electric and magnetic field components resulting in hybrid modes. Hence the analysis of optical waveguide is more complex than metallic hollow waveguide analysis.

- Depending on the large E-field, the hybrid modes are HE or EH modes. The two lowest order does are HE_{11} and TE_{01}.

Overview of Modes

- The order states the number of field zeros across the guide. The electric fields are not completely confined within the core i.e. they do not go to zero at core-cladding interface and extends into the cladding. The low order mode confines the electric field near the axis of the fiber core and there is less penetration into the cladding. While the high order mode distribute the field towards the edge of the core fiber and penetrations into the cladding. Therefore cladding modes also appear resulting in power loss.

- In leaky modes the fields are confined partially in the fiber core attenuated as they propagate along the fiber length due to radiation and tunnel effect.

- Therefore in order to mode remain guided, the propagation factor $\beta$ must satisfy the condition

$$n_2k < \beta < n_1k$$

where, $n_1$ = Refractive index of fiber core

$n_2$ = Refractive index of cladding

$k = \text{Propagation constant} = \frac{2\pi}{\lambda}$

- The cladding is used to prevent scattering loss that results from core material discontinuities. Cladding also improves the mechanical strength of fiber core and reduces surface contamination. Plastic cladding is commonly used. Materials used for fabrication of optical fibers are silicon dioxide (SiO_2), boric oxide-silica.

Summary of Key Modal Concepts

- Normalized frequency variable, $V$ is defined as
where, 

\( a = \) Core radius

\( \lambda = \) Free space wavelength

\[ V = \frac{2\pi a}{\lambda} NA \]  

Since \( (n_1^2 - n_2^2)^{1/2} = NA \)  

\[ (1.7.2) \]

- The total number of modes in a multimode fiber is given by

\[
M = \frac{1}{2} \left( \frac{2\pi a}{\lambda} \right)^2 (n_1^2 - n_2^2)
\]

\[
M = \frac{1}{2} \left[ \frac{2\pi a}{\lambda} \cdot NA \right]^2 = \frac{[V]^2}{2}
\]

\[
M = \frac{1}{2} \left[ \frac{\pi d}{\lambda} \cdot NA \right]^2 \quad \text{‘d’ is core diameter} \quad \cdots (17.3)
\]

**Example 1.7.1**: Calculate the number of modes of an optical fiber having diameter of 50 µm, \( n_1 = 1.48, n_2 = 1.46 \) and \( \lambda = 0.82 \) µm.

**Solution**: 

\( d = 50 \) µm

\( n_1 = 1.48 \)

\( n_2 = 1.46 \)

\( \lambda = 0.82 \) µm

\[ NA = (n_1^2 - n_2^2)^{1/2} \]

\[ NA = (1.48^2 - 1.46^2)^{1/2} \]

\[ NA = 0.243 \]
Number of modes are given by,

\[ M = \frac{1}{2} \left[ \frac{\pi d}{\lambda} \cdot NA \right]^2 \]

\[ M = \frac{1}{2} \left[ \frac{\pi (50 \times 10^{-6})}{0.82 \times 10^{-6} \times 0.243} \right]^2 \]

\[ M = 1083 \quad \text{...Ans.} \]

**Example 1.7.2**: A fiber has normalized frequency \( V = 26.6 \) and the operating wavelength is 1300nm. If the radius of the fiber core is 25 \( \mu \text{m} \). Compute the numerical aperture.

Solution :

\[ V = 26.6 \]

\[ \lambda = 1300 \text{ nm} = 1300 \times 10^{-9} \text{ m} \]

\[ a = 25 \mu \text{m} = 25 \times 10^{-6} \text{ m} \]

\[ V = \frac{2\pi a}{\lambda} \cdot NA \]

\[ NA = V \cdot \frac{\lambda}{2\pi a} \]

\[ NA = 26.6 \cdot \frac{1300 \times 10^{-9}}{2\pi \times 25 \times 10^{-6}} \]

\[ NA = 0.220 \quad \text{...Ans.} \]

**Example 1.7.3**: A multimode step index fiber with a core diameter of 80 \( \mu \text{m} \) and a relative index difference of 1.5 \% is operating at a wavelength of 0.85 \( \mu \text{m} \). If the core refractive index is 1.48, estimate the normalized frequency for the fiber and number of guided modes.

[July/Aug.-2008, 6 Marks]

**Solution**: Given: MM step index fiber, \( 2a = 80 \mu \text{m} \)

\[ \therefore \text{Core radius} \ a = 40 \mu \text{m} \]
Relative index difference, $\Delta = 1.5\% = 0.015$

Wavelength, $\lambda = 0.85\mu m$

Core refractive index, $n_1 = 1.48$

Normalized frequency, $V =$?

Number of modes, $M =$?

Numerical aperture

$$NA = n_1 (2\Delta)^{1/2}$$

$$= 1.48 (2 \times 0.015)^{1/2}$$

$$= 0.2563$$

Normalized frequency is given by,

$$V = \frac{2\pi a}{\lambda} NA$$

$$V = \frac{2\pi \times 40}{0.85} \times 0.2563$$

$$V = 75.78$$

... Ans.

Number of modes is given by,

$$M = \frac{V^2}{2}$$

$$M = \frac{(75.78)^2}{2} = 2871.50$$

... Ans.

**Example 1.7.4:** A step index multimode fiber with a numerical aperture of 0.20 supports approximately 1000 modes at an 850 nm wavelength.

i) What is the diameter of its core?

ii) How many modes does the fiber support at 1320 nm?

iii) How many modes does the fiber support at 1550 nm?  

[Jan./Feb.-2007, 10 Marks]

**Solution:** i) Number of modes is given by,
\[
M = \frac{1}{2} \left[ \frac{\pi a}{\lambda}.NA \right]^2
\]

\[
1000 = \frac{1}{2} \left[ \frac{\pi a}{850 \times 10^{-9}} \times 0.20 \right]^2
\]

\[
2000 = 5.464 \times a^2
\]

\[ a = 60.49 \mu m \quad \text{... Ans.} \]

ii)

\[
M = \frac{1}{2} \left[ \frac{\pi \times 60.49 \times 10^{-6}}{1320 \times 10^{-9}} \times 0.20 \right]^2
\]

\[ M = (14.39)^2 = 207.07 \quad \text{... Ans.} \]

iii)

\[
M = \frac{1}{2} \left[ \frac{\pi \times 6.49 \times 10^{-6}}{1320 \times 10^{-9}} \times 0.20 \right]^2
\]

\[ M = 300.63 \quad \text{... Ans.} \]

**Wave Propagation**

**Maxwell’s Equations**

Maxwell’s equation for non-conducting medium:

\[
\nabla \times E = -\frac{\partial B}{\partial t}
\]

\[
\nabla \times H = -\frac{\partial D}{\partial t}
\]

\[
\nabla \cdot D = 0
\]

\[
\nabla \cdot B = 0
\]

Where,

E and H are electric and magnetic field vectors.
D and B are corresponding flux densities.

- The relation between flux densities and filed vectors:
  \[
  D = \varepsilon_0 E + P \\
  B = \mu_0 H + M
  \]

Where,

\( \varepsilon_0 \) is vacuum permittivity.

\( \mu_0 \) is vacuum permeability.

P is induced electric polarization.

M is induced magnetic polarization (M = 0, for non-magnetic silica glass)

- P and E are related by:
  \[
  P(r, t) = \varepsilon_0 \int_{-\infty}^{\infty} X (r, t - t') E (r, t') dt'
  \]

Where,

X is linear susceptibility.

- Wave equation:
  \[
  \nabla \times \nabla \times E = \frac{1}{c^2} \frac{\partial^2 E}{\partial t^2} - \mu_0 \frac{\partial^2 P}{\partial t^2}
  \]

Fourier transform of E (r, t)

\[
\hat{E} (r, \omega) = \int_{-\infty}^{\infty} E (r, t)e^{i\omega t} dt
\]

\[
\nabla \times \nabla \times \hat{E} = -\varepsilon (r, \omega) \frac{\omega^2}{c^2} \hat{E}
\]

Where,

\[
\varepsilon = \left( n + \frac{i\alpha c}{2\omega} \right)^2
\]
n is refractive index.

\[ n = \sqrt{1 + \frac{R_e \chi}{c}} \]

\[ \alpha = \left( \frac{\omega}{n \cdot c} \right) I_m \chi \]

- Both \( n \) and \( \alpha \) are frequency dependent. The frequency dependence of \( n \) is called as chromatic dispersion or material dispersion.
- For step index fiber,

\[ \nabla \times \nabla \times \mathbf{E} = \nabla \left( \nabla \cdot \mathbf{E} \right) - \nabla^2 \mathbf{E} = -\nabla^2 \mathbf{E} \]

Fiber Modes

**Optical mode:** An optical mode is a specific solution of the wave equation that satisfies boundary conditions. There are three types of fiber modes.

a) Guided modes
b) Leaky modes
c) Radiation modes

- For fiber optic communication system guided mode is used for signal transmission.

Considering a step index fiber with core radius ‘a’.

The cylindrical co-ordinates \( \rho, \phi \) can be used to represent boundary conditions.

\[ \frac{\partial^2 E_z}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial E_z}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 E_z}{\partial \phi^2} + \frac{\partial^2 E_z}{\partial z^2} + n^2 k_0^2 E_z = 0 \]

- The refractive index ‘\( n \)’ has values

\[ n = \begin{cases} n_1; & \rho \leq a \\ n_2; & \rho > a \end{cases} \]

- The general solutions for boundary condition of optical field under guided mode is infinite at \( \rho = 0 \) and decay to zero at \( \rho = \infty \). Using Maxwell’s equation in the core region.

\[ E_\rho = \frac{i}{\rho^2} \left( \beta \frac{\partial E_z}{\partial \rho} + \mu_0 \frac{\omega}{\rho} \frac{\partial H_z}{\partial \phi} \right) \]
The cut-off condition is defined as –

\[ V = k_0 a \sqrt{(n_1^2 - n_2^2)} \]

\[ V = \left( \frac{2\pi}{\lambda} \right) a n_1 \sqrt{2\Delta} \]

It is also called as normalized frequency.

**Graded Index Fiber Structure**

- The refractive index of graded index fiber decreases continuously towards its radius from the fiber axis and that for cladding is constant.
- The refractive index variation in the core is usually designed by using power law relationship.

\[
n(r) = \begin{cases} 
  n_1 \left[ 1 - 2\Delta \left( \frac{r}{a} \right) \right]^\alpha, & \text{when } 0 \leq r \leq a \\
  n_2 (1 - 2\Delta)^\alpha \approx n_1 (1 - \Delta) = n_2, & \text{when } r \geq a 
\end{cases} \quad \ldots (1.7.4)
\]

Where,

- \( r \) = Radial distance from fiber axis
- \( a \) = Core radius
- \( n_1 \) = Refractive index core
- \( n_2 \) = Refractive index of cladding
- \( \alpha \) = The shape of the index profile

For graded index fiber, the index difference is given by,
In graded index fiber the incident light will propagate when local numerical aperture at distance \( r \) from axis, \( \text{NA} \) is axial numerical aperture \( \text{NA}(0) \). The local numerical aperture is given as,

\[
\Delta = \frac{n_1^2 - n_2^2}{2n_1^2}
\]

\[
\Delta = \frac{n_1 - n_2}{n_1}
\]

- The axial numerical aperture \( \text{NA}(0) \) is given as,

\[
\text{NA}(r) = \begin{cases} 
[n^2(r) - n_2^2]^{1/2} \approx \text{NA}(0) \sqrt{1 - \left(\frac{r}{a}\right)^2}, & \text{for } r \leq a \\
0, & \text{for } r > a
\end{cases}
\]

- The variation of NA for different values of \( \alpha \) is shown in Fig. 1.7.1.

Hence Na for graded index decreases to zero as it moves from fiber axis to core-cladding boundary.

- The variation of NA for different values of \( \alpha \) is shown in Fig. 1.7.1.
1. The number of modes for graded index fiber in given as,

\[ M = \frac{\alpha}{\alpha + 2} a^2 k^2 n_1^2 \Delta \]  

\( \ldots (1.7.6) \)

1.8 Single Mode Fibers

- Propagation in single mode fiber is advantageous because signal dispersion due to delay differences amongst various modes in multimode is avoided. Multimode step index fibers cannot be used for single mode propagation due to difficulties in maintaining single mode operation. Therefore for the transmission of single mode the fiber is designed to allow propagation in one mode only, while all other modes are attenuated by leakage or absorption.
- For single mode operation, only fundamental LP\(_{01}\) mode many exist. The single mode propagation of LP\(_{01}\) mode in step index fibers is possible over the range.

\[ 0 \leq V < 2405 \]

- The normalized frequency for the fiber can be adjusted within the range by reducing core radius and refractive index difference < 1%. In order to obtain single mode operation with maximum V number (2.4), the single mode fiber must have smaller core diameter than the equivalent multimode step index fiber. But smaller core diameter has problem of launching light into the fiber, jointing fibers and reduced relative index difference.
- Graded index fibers can also be sued for single mode operation with some special fiber design. The cut-off value of normalized frequency \(V_c\) in single mode operation for a graded index fiber is given by,

\[ V_c = 2.405 \left( 1 + \frac{2}{\alpha} \right)^{1/2} \]

**Example 1.8.1:** A multimode step index optical fiber with relative refractive index difference1.5\% and core refractive index 1.48 is to be used for single mode operation. If the operating wavelength is 0.85\(\mu\)m calculate the maximum core diameter.

Solution: Given

\[ n_1 = 1.48 \]

\[ \Delta = 1.5 \% = 0.015 \]
\[ \lambda = 0.85 \, \mu m = 0.85 \times 10^{-6} \, m \]

Maximum V value for a fiber which gives single mode operations is 2.4.

Normalized frequency (V number) and core diameter is related by expression,

\[ V = \frac{2\pi}{\lambda} a \, (NA) \]

\[ V = \frac{2\pi}{\lambda} a \, n_1 (2\Delta)^{\frac{1}{2}} \]

\[ a = \frac{V \lambda}{2\pi n_1 (2\Delta)^{\frac{1}{2}}} \]

\[ a = \frac{2.4 \times (0.85 \times 10^{-6})}{2\pi \times (1.48) \times (0.03)^{\frac{1}{2}}} \]

\[ a = 1.3 \, \mu m \]

Maximum core diameter for single mode operation is 2.6 \( \mu m \).

**Example 1.8.2:** A GRIN fiber with parabolic refractive index profile core has a refractive index at the core axis of 1.5 and relative index difference at 1%. Calculate maximum possible core diameter that allows single mode operations at \( \lambda = 1.3 \, \mu m \).

**Solution: Given:**

\[ n_1 = 1.5 \]

\[ \Delta = 1 \% = 0.01 \]

\[ \lambda = 1.3 \, \mu m = 1.3 \times 10^{-6} \, m \]

for a GRIN

Maximum value of normalized frequency for single mode operation is given by,

\[ V = 2.4 \left( 1 + \frac{2\Delta}{\infty} \right)^{\frac{1}{2}} \]
\[ V = 2.4 \left( 1 + \frac{2}{2} \right)^{\frac{1}{3}} \]

\[ V = 2.4 \sqrt{2} \]

Maximum core radius is given by expression,

\[ a = \frac{V\lambda}{2\pi n_1 (2\Delta)^{\frac{1}{2}}} \]

\[ a = \frac{24\sqrt{2} \times 1.3 \times 10^{-6}}{2\pi \times 1.5 \times (0.02)^{\frac{1}{2}}} \]

\[ a = 3.3 \text{ \(\mu\)m} \]

\[ \therefore \text{Maximum core diameter which allows single mode operation is 6.6 \(\mu\)m.} \]